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# EQUATIONS OF ISOTROPIC DEFORMATION OF GAS-SATURATED MATERIALS WITH ALLOWANCE FOR LARGE STRAINS OF SPHERICAL PORES

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We will examine a composite medium consisting of a homogeneous isotropic matrix and spherical pores saturated with gas. The character of location of the pores is assumed to be statistically uniform. The effective-field method was used in [1-3] to obtain equations of state of gas-saturated porous media with the assumption of small strains of the pores and the medium as a whole [1]. In the case of large general strains, it is natural to examine methods of solution involving the use of successive approximations [4], such as was done in in an examination of composite media by the method of conditional functions [5]. The latter method is based on the assumption that the stress field is uniform within each component of the composite – an assumption which leads to large errors in evaluating the effective parameters of linearly elastic media compared to the effective-field method [1, 2]. The authors of [6, 7] analyzed arbitrarily large strains for the special case of isotropic deformation of a material with spherical pores and an incompressible matrix, using a cellular model to perform the analysis. Here, we solve a similar problem with allowance for the effect of gas pressure in the pores, and we make use of the ideas behind the effective-field method [1, 2] in doing so. The usefulness of this method has been proven in studies of linear problems for micro-inhomogeneous media.

1. Physical Model. In a number of cases of practical importance, it is of interest to study the volumetric deformation of rubber-like materials with a low ($\leqslant 1\%$) porosity. For the sake of determinateness, we will describe the strain properties of the matrix with a Mooney potential [4]. The authors of [1] showed that in linear problems of gas-saturated porous media, the effects of binary interaction of inclusions are unimportant for spherical pores in an incompressible matrix in the case of low porosity. Here, the effective bulk modulus is determined by the solution of the linearly elastic problem of a single inclusion in a matrix with a certain effective stress field specified at infinity. Thus, it is acceptable to make use of the cellular model in [6, 7]. This model presumes equivalent strain properties for a porous medium and a thick-walled spherical shell and equality of the ratio of the volumes of the pore and spherical element to the porosity of the composite medium being modeled. Here, we will use the positive ideas behind the effective-field method and we will place the spherical element in a matrix with a prescribed effective stress field at infinity which differs from the acting stress field. We find the parameters of this field by the self-consistent effective-field method [1, 3]. The method makes it unnecessary to postulate the relationship between the relative dimensions of the spherical element and the porosity of

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the medium, but it presumes that this porosity is small enough so that in the event of a large strain of the surface of the pores, the strains of the surface of the spherical element will remain small.

2. Effective-Field Method in Linear Problems of Micro-Inhomogeneous Media. We present certain relations which follow from [1, 2] and which will prove useful in the discussions to follow. In a linearly elastic medium with the modulus $L_0$, let there be distributed a Poisson set $X = (x_k, r_k, V_k)$ of spherical pores $v_k$ with centers at $x_k$. The pores have the radii $r_k$ and characteristic functions $V_k$. The pressure of the gas in the pores is p. Then the equation of state of the material has the form

$$\sigma(x) = (1 - V(x))L_0\varepsilon(x) - V(x)q. \tag{2.1}$$

Here, $q = p\delta_{ij}$ is a divalent tensor; $V(x) = \bigcup_k V_k(x)$; σ and ε are the stress and strain tensors. In the case of isotropy of the components

$$L_0 = (3k_0,\ 2\mu_0) = 3k_0N_1 + 2\mu_0N_2,$$
$$N_1 = (1/3)\delta_{ij}\delta_{kl},\ N_2 = (1/3)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - (2/3)\delta_{ij}\delta_{kl}).$$

Equation (2.1) differs from the equation adopted in [8]

$$\sigma(x) = (1 - V(x))L_0(x)\varepsilon(x) + (L_2\varepsilon(x) - q_0)V(x), \tag{2.2}$$

where $L_2 = (3k_2, 2\mu_2)$ is the isotropic elastic modulus of the pore phase; $\mu_2 = 0$ is the shear modulus; $k_2 \neq 0$; $q_0 = p_0\delta_{ij}$; $p_0$ is the gas pressure in the undeformed initial material. Use of the current pressure p (2.1) rather than the initial pressure $p_0$ (2.2) leads to "disappearance" of the bulk modulus of the pore phase and simplification of the subsequent calculations.

Insertion of (2.1) into the equilibrium equation div σ = 0 and subsequent solution of the resulting relations by the effective-field method [1] make it possible to evaluate the effective parameters $L_*$ and $\varepsilon_q$ of the equation of the macroscopic state

$$\langle\sigma\rangle = L_*(\langle\varepsilon\rangle - \varepsilon_q),\ \varepsilon_q = (L_*^{-1} - L_0^{-1})_q \tag{2.3}$$

(<...> denotes the operation of averaging over the micro-volume of the porous medium). In the case of spherical pores of one size, we obtain [1] $L_* = (3k_*, 2\mu_*)$, where

$$3k_* = (4\mu_0/c)\,[1 - (29/24)\,c]; \tag{2.4}$$

$$2\mu_* = 2\mu_0\,\{1 + (5/3)\,c\,[1 - (35/24)\,c]^{-1}\}^{-1}. \tag{2.5}$$

Here, the terms in the brackets, describing binary interaction of inclusions, are unimportant in the case of low porosity c = <V> compared to the terms obtained from the solution of the problem of a single pore in a matrix with an effective field $\bar{\varepsilon}$ assigned at infinity [1]. In the present case, this field is equal to

$$\bar{\varepsilon} = D^{\varepsilon}(\langle\varepsilon\rangle + J),\ D^{\varepsilon} = ([1 - (29/24)c]^{-1},\ [1 - (35/24)c]^{-1}),\ J = ((5/24)cp,\ 0). \tag{2.6}$$

In fact, with a porosity c = 1%, the bulk modulus $k_*$ evaluated from (2.4) with and without allowance for binary interaction of the inclusions is equal to $131.7\mu_0$ and $133.3\mu_0$, respectively. The shear modulus $\mu_*$ in the case of low porosity depends slightly on c; with a decrease in c by a factor of 4 (from 2 to 0.5%), $\mu_*$ increases by only 2.5% – from $0.97\mu_0$ to $0.99\mu_0$.

Let us analyze a problem we will need later on – the problem of the deformation of a composite medium consisting of a linearly elastic matrix with a modulus $L_0$ and a Poisson set $X = (x_k, r_k', V_k, L_1^{(k)}, \alpha^{(k)})$ of spherical inclusions $v_k'$ with centers at $x_k$, radii $r_k'$, characteristic functions $V_k$, moduli $L_0 + L_1^{(k)}$, and the parameter $\alpha^{(k)}$. Thus, the local equation of state has the form

$$\sigma(x) = L(x)(\varepsilon(x) - \alpha(x)). \tag{2.7}$$

Here, $\alpha(x) = 0$, $L(x) = L_0$ at $V(x) = 0$ and $L(x) = L_0 + L_1^{(k)}$, $\alpha(x) = \alpha^{(k)}$ at $x \in v_k$. Equation (2.7) formally coincides with the equation of thermoelasticity of a composite medium [2] with a zero coefficient of linear expansion of the matrix. To evaluate the effective parameters of the composite medium $L_*$, $\alpha_*$ in the equation

$$\langle\sigma\rangle = L_*(\langle\varepsilon\rangle - \alpha_*) \tag{2.8}$$

this coincidence allows us to make use of the effective-field method [2]

$$L_* = L_0 (I + L_0 D \langle BM_1 V\rangle)^{-1}, \quad \alpha_* = D\langle B\alpha V\rangle, \tag{2.9}$$

where the study [2] presented the following expression for the tensor of the stress concentration B for an isolated inclusion in an infinite matrix and the tensor of the stress concentration D describing the effect of the surrounding inclusions: $M_1(x) \equiv (L_0 + L_1^{(k)})^{-1} - L_0^{-1}$ at $x \in V_k$.

3. Calculation of the Spherical Element. We will examine a thick-walled spherical shell with internal and external radii in the undeformed state $R_1$ and $R_2$. The strain properties of the material are described by a Mooney potential with the constants $a_1$, $a_2$

$$W = a_1(I_1 - 3) + a_2(I_2 - 3) \tag{3.1}$$

[$I_i$ (i = 1, 2, 3) are invariants associated with the Lagrangian strain tensor; under conditions of central-symmetrical deformation, these invariants are connected with the extensions $\lambda$ by the relations $I_1 = 3\lambda^2$, $I_2 = 3\lambda^4$, $I_3 = \lambda^6$]. The extension parameter $\lambda$ determines the relation describing the distances of points of the element to the center of the shell before and after deformation $R = \lambda^{-1}r$. For the internal and external surfaces of the shell, we obtain $r_1 = \lambda_1 R_1$ and $r_2 = \lambda_2 R_2$.

In the case of loading of the shell by internal and external pressures $p_1$ and $p_2$, the solution for large strains of the spherical element is known [6, 9]:

$$-p_2 = a_1\left[1/\lambda_2^4 + 4/\lambda_2 - (1/\lambda_1^4 + 4\lambda_1)\right] + 2a_2\left[1/\lambda_2^2 - 2\lambda_2 - (1/\lambda_1^2 - 2\lambda_1)\right] - p_1. \tag{3.2}$$

For an incompressible material, we have the equality $r_1^3 - R_1^3 = r_2^3 - R_2^3$. Then, using the parameter $\gamma = R_1^3/R_2^3$ characterizing the relative fraction of the pore volume in the undeformed spherical element, we find

$$\lambda_1^3 = (\lambda_2^3 - 1)/\gamma + 1. \tag{3.3}$$

Equations (3.2) and (3.3) allows us to use assigned values of $p_1$ and $p_2$ to find $\lambda_1$ and $\lambda_2$. We henceforth choose $\gamma$ to be small enough so that $\xi$ is small in the expression $\lambda_2 = 1 + \xi$ and $\lambda_1^3 = 3\xi/\gamma + 1$. Then the spherical element can be replaced by a linearly elastic sphere whose strain properties are described by Eq. (2.7) with the moduli

$$L(x) = L^e = (3k^e, 2\mu^e), \quad \alpha(x) = \alpha^e = 3\xi^e\delta_{ij}, \tag{3.4}$$

where $\xi^e$ is found from the solution of (3.2) and (3.3) with $p_2 = 0$ and $k^e = p_2/(1 + \xi^e - \lambda_2)$; $\mu^e = \mu_*$; due to the small effect of porosity on $\mu_*$ with small c, we used Eq. (2.5), obtained with small strains of the pores, for $\mu^e$.

4. Evaluation of the Effective Parameters of the Medium. In accordance with the physical model of a gas-saturated porous medium, we will assume that its isotropic deformation is equivalent to the isotropic deformation of a composite medium consisting of a matrix with the modulus $L_0 = (3k_0, 2\mu_0)$, $k_0 = \infty$, $\mu_0 = 2(a_1 + a_2)$ and a Poisson set of spherical inclusions with the modulus $L^e$ and the parameter $\alpha^e$ (3.4). For the sake of determinateness, we will examine inclusions of one size with the degree of fullness $c^e = c_0/\gamma$ ($c_0$ is the porosity in the undeformed state). Then the equation of isotropic deformation is described by Eqs. (2.8) and (2.9). In the latter equations, the tensors B and D are expressed in a known manner in terms of the quantities $L_0$, $L^e$, and $c^e$ [2].

The parameter $\alpha_*$ in (2.8) depends on the gas pressure $p_1$, which in turn is determined by the deformation of the pore phase. In the case where the empirically established mean-volume concentration of gas w in the macro-region is assigned, then in accordance with the laws discovered by Henry and Mendeleev–Clapeyron [1]

$$p_1 = w\left[(1 - c\lambda_1^3)\Gamma + c\lambda_1^3\mu'/R'T\right]. \tag{4.1}$$

Here, the first term in brackets with the Henry constant $\Gamma$ describes the contribution of the mean concentration of gas dissolved in the solid phase. The second term accounts for the presence of gas with a molecular weight $\mu'$ at the temperature T in the pore phase; R' is the gas constant. Equation (4.1) obviously generalizes to a mixture of gases.

It should be noted that the individual assumptions made here are not fundamental and can be made less restrictive. In fact, one important assumption is that the strains of the external surface of the spherical element are small, which allowed us to reduce the problem to a linear problem (2.7) and solve it by the effective-field method [2]. In the case of large

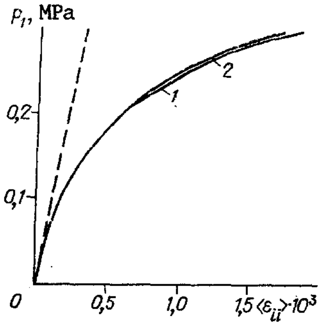


Fig. 1

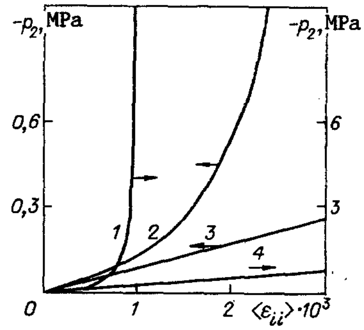


Fig. 2

strains of the pores, this assumption makes it necessary to select a sufficiently small value of $\gamma$ (3.3) — which can be done only in the case of low porosity. The assumptions made regarding the incompressibility of the matrix and the use of the Mooney potential (3.1) are required only in an analytical solution of the problem for a spherical element (3.2), (3.3). In the general case, we can examine any elastic potential of strain energy W with a numerical solution of the problem for the spherical element (3.4).

As an example, we will examine different cases of isotropic loading of a gas-saturated porous medium. Let $\langle\sigma\rangle = 0$. We will find the macroscopic strain of the medium $\langle\varepsilon\rangle$ and the porous phase $c(\lambda_1^3 - 1)$ with different values of $p_1$. We take values characteristic of raw rubber: $a_1 = 0.1$ MPa, $a_2 = 0.01$ MPa [6], and $c = 10^{-3}$. Curves 1 and 2 in Fig. 1 (for $\gamma = 10^{-2}$ and $2\cdot 10^{-3}$) were calculated from nonlinear theory (2.8), (3.4), while the dashed curve was calculated from linear relations (2.3), (2.4). The small difference between the curves is attributable to the self-consistency of the estimates of D and B (2.9) obtained by the effective-field method. We can use Fig. 1 to evaluate the difference in the results calculated by means of the linear and nonlinear models for moderate values of $p_1$, while the additional use of Eq. (4.1) makes it possible to find the value of the mean-volume gas concentration w necessary for the given loading regime. Curves 1, 4 and 2, 3 in Fig. 2 show the volumetric deformation of the medium with $p_1 = 0$, $c = 10^{-3}$, and $c = 5\cdot 10^{-3}$, respectively. Lines 3 and 4 were calculated from linear theory (2.3), (2.4), while lines 1 and 2 were calculated from the nonlinear theory. With a negative hydrostatic stress, the curves $\langle\sigma_{ii}\rangle = \langle\sigma_{ii}\rangle(\langle\varepsilon_{ii}\rangle)$ have a vertical asymptote. At small c, this asymptote is equal to $\langle\varepsilon_{ii}\rangle = c_0$, and it approaches the y axis $\langle\varepsilon_{ii}\rangle = 0$ with a decrease in $c_0$. There is no vertical asymptote with isotropic expansion, and even at $c = 10^{-3}$ isotropic expansion of the medium may exceed the value $\langle\varepsilon_{ii}\rangle = 0.1$. Since a material always contains a certain number of pores, it follows that perfectly isotropic inextensible materials do not exist. Thus, there arises the question of the validity of using the Mooney potential in the region of large isotropic extensions. Moreover, the parameters of different elastic potentials of isotropically deformable materials are generally determined in hydrostatic compression [10]. In light of the above analysis, the use of these parameters in the region of large hydrostatic tension would appear to be incorrect.

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# ELASTOPLASTIC PROPERTIES OF MULTICOMPONENT COMPOSITES

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Predicting the inelastic properties of materials with random discontinuities is one of the most important current developments in the mechanics of deformable solids. Modeling the macroscopic governing relations and calculating the effective characteristics of such media in many cases permits satisfactory estimation of the strain properties, limiting state, and load-carrying capacity of structural elements made of composites, powders, and other types of structural materials. The macroscopic behavior of multicomponent rigid-plastic and elastoplastic composites was examined in [1, 2] within the framework of flow theory.

Here, we examine the use of the method of generalized singular approximation of the theory of random fields to describe small elastoplastic strains of composite materials with an arbitrary number of constituents. A similar problem was solved in a correlation approximation in [3, 4].

Let a micro-inhomogeneous medium occupying a volume V bounded by the surface S be composed of n different elastoplastic constituents connected to each other with ideal adhesion. The governing relations for the material of each constituent are given by the equations

$$s_{ij} = 2\mu_s(\varepsilon_{kl})e_{ij},\ \sigma_{pp} = 3K_s\varepsilon_{pp}\ (s = 1, 2, \ldots, n). \quad (1)$$

Here, $s_{ij} = \sigma_{ij} - (1/3)\delta_{ij}\sigma_{pp}$; $e_{ij} = \varepsilon_{ij} - (1/3)\delta_{ij}\varepsilon_{pp}$; $\sigma_{ij}$, $\varepsilon_{ij}$ are components of the stress and strain tensors; $\mu_s(\varepsilon_{kl})$ is the shear modulus of plasticity; $K_s$ is the bulk modulus of the s-th constituent ($K_s$ = const).

The structure of such a composite can be described by a set of random indicator functions of the coordinates $\varkappa_1(\mathbf{r}), \varkappa_2(\mathbf{r}), \ldots, \varkappa_n(\mathbf{r})$. Meanwhile, each function $\varkappa_s(\mathbf{r})$ is equal to unity on the set of points of the s-th constituent and is equal to zero outside this set. Using these functions, we can write the local governing equations (1) in the form

$$s_{ij}(\mathbf{r}) = 2\sum_{s=1}^{n} \mu_s(\varepsilon_{kl}(\mathbf{r}))\varkappa_s(\mathbf{r})e_{ij}(\mathbf{r}),\ \sigma_{pp}(\mathbf{r}) = 3\sum_{s=1}^{n} K_s\varkappa_s(\mathbf{r})\varepsilon_{pp}(\mathbf{r}). \quad (2)$$

All of the functions $\varkappa_s(\mathbf{r})$ of stress and strain are presumed to be statistically uniform and ergodically random fields, and their mathematical expectations are replaced by the mean values over the volumes of the constituents $V_s$ and over the total volume of the composite $V = \sum_{s=1}^{n} V_s$: $\langle f\rangle_s = \frac{1}{V_s}\int_{V_s} f(\mathbf{r})\,d\mathbf{r}$, $\langle f\rangle = \frac{1}{V}\int_V f(\mathbf{r})\,d\mathbf{r}$.

Establishing the macroscopic governing equations and effective constants of such a composite mean determining the relation between the macrostresses $\langle\sigma_{ij}\rangle$ and the macrostrains $\langle\varepsilon_{ij}\rangle$. The general form of this relation is expressed in the present case by the formula

$$\langle\sigma_{ij}\rangle = E^*_{ijkl}(\langle\varepsilon_{mn}\rangle)\langle\varepsilon_{kl}\rangle \quad (3)$$

($E^*_{ijkl}(\langle\varepsilon_{mn}\rangle)$ are components of the fourth-rank tensor of the plastic moduli). Here and below, an asterisk denotes an effective value of a quantity.

To derive Eqs. (3), it is necessary to statistically average the system of equations describing the deformation of an inhomogeneous medium. This system consists of (2), the equilibrium equations

$$\sigma_{ip,p}(\mathbf{r}) = 0 \quad (4)$$

and the Cauchy formulas

$$2\varepsilon_{ij}(\mathbf{r}) = u_{i,j}(\mathbf{r}) + u_{j,i}(\mathbf{r}), \quad (5)$$

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